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ON EDGE OPEN PACKING SETS OF GRAPHS

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Abstract: A nonempty subset of the edge set of a graph G is called an edge open packing set if no two edges of it have a common edge. The edge open packing number $\rho_e^o(G)$ of a graph G is the maximum number of edges in an edge open packing set. In this paper, a number of results are presented concerning lower and upper bounds of $\rho_e^o(G)$ for graphs such as trees, split graphs and unicyclic graphs. Some open problems are proposed.

Keywords and Phrases: Open packing number, edge open packing number, split graphs, unicyclic graphs.

2020 Mathematics Subject Classification: 05C12, 05C70, 05C75.

1. Introduction

Here we concentrate only on graphs that are simple, finite, undirected and connected. For basic notations and terminology one can refer to Chartrand G. and Lesniak [1].

A primary reason for our interest towards this topic is because of the results in Meir A. and Moon J. W. [4]. In Sahul Hamid I. and Saravanakumar S. [10, 11], the authors investigated the effect of change in the value of the open packing number when a vertex or an edge is removed. In Sahul Hamid I. and Saravanakumar S. [12], the authors characterized the graphs G with open packing number value as $n-2$, $n-\omega(G)$ and $n-\Delta(G)$, where $n=|V(G)|$, $\omega(G)$ is the clique number of G and $\Delta(G)$ is the maximum degree of G. They also made some discussion about open packing number for split graphs. An interesting application about open packing number was obtained in Gao Y. et al. [3] by computing it for certain chemical graphs. Babak Samadi et al. in [7] determined the open packing and k-limited packing numbers and settled the task of characterizing all connected extremal graphs on n vertices with open packing number value equal to the quotient of n and $\delta(G)$. The authors in Mojdeh D. A. et al. [8] gave upper bounds on packing and open packing numbers of triangle free graphs and determined the extremal graph characterization results of Nordhaus-Gaddum type for packing numbers. In Mohammadi M. et al. [5], the authors determined a new characterization result concerning open packing number and total domination number of graphs of order n, $\delta(G)$, at least two with $\frac{n}{2}=\rho^o(G)=\gamma_t(G)$. In Vaidya S. K. et al. [14, 15], the authors determined the exact values of packing and open packing numbers for certain families of graphs such as triangular snakes. In Mojdeh D. A. et al. [6], the authors determined a complete solution with respect to lexicographic and rooted product of graphs and developed a number of lower and upper bounds for packing number and open packing number. In Raja Chandrasekar K. [9], the authors determined the exact value of open packing numbers of certain perfect graphs such as split graphs, $\{P_4, C_4\}$-free graphs etc.

In $G=(V,E)$, two edges e_1 and e_2 are said to have a *common edge* if there exists an edge $e \in E(G)$ different from e_1 and e_2 such that e joins a vertex of e_1 to a vertex of e_2 in G. That is, $\langle e_1, e, e_2\rangle$ is either P_4 or K_3 in G. A set $D \subseteq E(G)$ with $D \neq \phi$ is called an *edge open packing set(EOPS)* of G if no two edges of D have a common edge in G. The maximum cardinality of an EOPS is called the *edge open packing number(EOPN)* of G and is denoted by $\rho_e^o(G)$. The EOPN was coined by the authors in Gayathri C. et al. [2]. Also, in [2], various bounds, characterization results and realization theorems on this number were obtained.

In this paper, we obtained several characterization results and developed certain bounds for $\rho_e^o(G)$. The following results are taken from Gayathri C. et al. [2] and are used to derive some results listed here.

Proposition 1.1. [2] *Let P_n be a path of size $m \geq 2$. Then*

$$\rho_e^o(P_n) = \begin{cases} \frac{m+2}{2} & if\ m \equiv 2 (mod\ 4) \\ \lceil \frac{m}{2} \rceil & otherwise. \end{cases}$$

Proposition 1.1. [2] *For the cycles C_n with size $m \geq 3$, we have*

$$\rho_e^o(C_n) = \begin{cases} \frac{m}{2} - 1 & if\ m \equiv 2 (mod\ 4) \\ \left\lfloor \frac{m}{2} \right\rfloor & otherwise. \end{cases}$$

Proposition 1.1. [2] $\rho_e^o(G) = 2$ *if and only if the following conditions are true*

(i) $2 \leq diam(G) \leq 4$

(ii) G *is* $K_{1,s}$*-free, where* $s \geq 3$ *and*

(iii) for any two non-adjacent edges $e_1 = uv$ *and* $e_2 = xy$ *such that* e_1 *and* e_2 *have no common edge in* G*, every vertex in* $V(G) \setminus \{u, v, x, y\}$ *is adjacent to at least two vertices in the set* $\{u, v, x, y\}$*.*

2. Some bounds and characterization Results on EOPN of Graphs

By $N_e(v)$, we mean the set of edges incident at the vertex v. Clearly $deg(v) = |N_e(v)|$. The vertex $v \in V(G)$ is said to be a *maximum degree vertex* of a graph G if $deg(v) = \Delta(G)$. $V_{max}(G) = \{u \in V(G) : deg(u) = \Delta(G)\}$. Set $N_i(v) = \{u : d(u, v) = i\}$.

Definition 2.1. *A vertex* v *is said to be a major independent vertex if*

(i) $v \in V_{max}(G)$

(ii) $N(v)$ *is an independent set.*

2.1. Connected Split Graphs

Definition 2.2. [13] *A graph* G *is said to be a split graph if the vertex set* $V(G)$ *can be partitioned into two non-empty sets* V_1 *and* V_2 *such that* $\langle V_1 \rangle$ *is complete and* $\langle V_2 \rangle$ *is totally disconnected. Here* (V_1, V_2) *is called a split partition of* G*.*

Remark 2.3. *Let* G *be a split graph with split partition* (V_1, V_2)*. If there exists a vertex* $v \in V_2$ *such that* v *is adjacent to every vertex of* V_1*, then* $(V_1 \cup \{v\}, V_2 - \{v\})$ *is also a split partition of* G*. Hence we may assume without loss of generality that* $|V_1| = \omega(G)$ *and every vertex of* V_2 *is nonadjacent to at least one vertex of* V_1*. Hence* $deg(v) \leq |\omega(G)| - 1$ *for all* $v \in V_2$ *and* $deg(v) \geq \omega(G)$ *for at least one* $v \in V_1$*. Hence there exists* $v \in V_1$ *such that* $deg(v) = \Delta(G)$*. Also any two edges in the clique* V_1 *have a common edge and hence any EOPS of* G *contains exactly one edge of the clique.*

Theorem 2.4. *Let* G *be a split graph with split partition* (V_1, V_2) *and let* $|V_1| = \omega(G)$*. Then* $\rho_e^o(G) \leq \Delta(G) - \omega(G) + 2$*. Also equality holds if and only if there exists an edge* uv *in* V_1 *such that* $deg(u) = \Delta(G)$ *and* $N(u) \cap N(v) \cap V_2 = \phi$*.*

Proof. Let D ba any EOPS in G. Since any two edges in $G[V_1]$ have a common edge, D contains exactly one edge uv from $G[V_1]$. If D contains two edges e_1 and e_2 incident with u and v respectively, then uv is a common edge of e_1 and e_2, which is a contradiction. Hence all edges of D are incident with one of the vertices u and v say v. Since $D \cap E(G[v_1]) = \{uv\}$, we have $|D| \leq deg(v) - \omega(G) + 2$. Hence $\rho_e^o(G) \leq \Delta(G) - \omega(G) + 2$. Suppose equality holds. Then $deg(v) = \Delta(G)$ and $D = \{uv\} \cup \{vv_i : v_i \in N(v) \cap V_2\}$. If u is adjacent to some v_i, then uv and vv_i has uv_i as a common edge, which is a contradiction. Hence $N(u) \cap N(v) \cap V_2 = \phi$. Conversely if $N(u) \cap N(v) \cap V_2 = \phi$, then D is an EOPS of G and $\rho_e^o(G) = |D| = \Delta(G) - \omega(G) + 2$.

2.2. Trees

Theorem 2.5. *Let T be a tree and let $k \geq 3$ be an integer. Then $\rho_e^o(T) \geq \sum_{v \in S} deg(v)$ where $S \subseteq V(T)$ is a k-packing set of maximum cardinality.*

Proof. By the definition of a k-packing set, we understand that such a set S possess the property that any two vertices of it are separated by a distance of at least $(k+1)$. Let E_1 be the set of all edges incident at the vertices of S. As S is a k-packing set, it is evident that any two edges of E_1 cannot have a common edge in E_1. Further, if any two edges $e_1 = (u_1, v_1)$, $e_2 = (u_2, v_2)$, where $u_1, u_2 \in S$ of E_1 possess a common edge in E_1, then $1 \leq d(u_1, u_2) \leq 3$, a contradiction. Hence $\rho_e^o(T)$ must be at least the sum of the degree of the vertices in S.

Preposition 2.6. *If $\rho_e^o(G) = 3$, then $diam(G) \leq 5$.*

Proof. When $diam(G) \geq 6$, then by Proposition 1.1, it is possible to construct an EOPS with four elements. As $\rho_e^o(G) = 3$, it follows that $diam(G) \leq 5$.

Preposition 2.7. *If G is a graph with at least one major independent vertex, then $\rho_e^o(G) \geq \Delta(G)$.*

Proof. Suppose that $u \in V(G)$ is a major independent vertex. Then it is easy to see that all the edges incident with u will form an EOPS. So it follows that $\rho_e^o(G) \geq \Delta(G)$.

Observation 2.8. *From Proposition 2.7, it follows readily that for any tree T, $\rho_e^o(T) \geq \Delta(T)$ as every maximum degree vertex in T is a major independent vertex.*

Theorem 2.9. *Let T be a tree with $\rho_e^o(T) = \Delta(T)$. Then $d(x, y) \leq 2$ for all $x, y \in V_{max}(T)$.*

Proof. If $\Delta(T) = 1$, then $G \cong K_2$. Now, let $\Delta(T) \geq 2$. Suppose that $d(x, y) = 3$ for some $x, y \in V_{max}(T)$. Let $P : (x, u, v, y)$ be a path in T. Then either the set $N_e(x) \cup N_e(y) \setminus \{xu\}$ or the set $N_e(x) \cup N_e(y) \setminus \{vy\}$ forms an EOPS of T. So $\rho_e^o(T) \geq |N_e(x)| + |N_e(y)| - 1 = 2\Delta(T) - 1$ implies that $\Delta(T) \leq 1$, a contradiction.

Further, if $d(x,y) \geq 4$ for some $x, y \in V_{max}(T)$, then the set $N_e(x) \cup N_e(y)$ is an EOPS of T. So $\rho_e^o(T) \geq |N_e(x)| + |N_e(y)| = 2\Delta(T)$. This is again a contradiction to $\rho_e^o(T) = \Delta(T)$. Hence $d(x,y) \leq 2$ for all $x, y \in V_{max}(T)$.

Theorem 2.10. *Let T be a tree with $\rho_e^o(T) = \Delta(T)$. Then $|V_{max}(T)| \leq 3$.*
Proof. Let $\rho_e^o(T) = \Delta(T)$. We claim that $|V_{max}(T)| \leq 3$. Suppose $|V_{max}(T)| \geq 4$. This implies that there exists at least four maximum degree vertices, say x, y, z, w in T. As six pair of distances among these vertices cannot exceed 2, we claim that all of them have exactly one common neighbor vertex x_1. Suppose there exists two common neighbor vertices x_1, x_2. Then $\langle\{x, y, z, w, x_1, x_2\}\rangle$ will contain a C_4, a contradiction. Note that $\langle\{x, y, z, w, x_1\}\rangle$ will contain $K_{1,4}$ as an induced subgraph. This implies that $deg(x) = deg(y) = deg(z) = deg(w) \geq 5$ as $x, y, z, w \in V_{max}(T)$. Now one can construct an EOPS with at least $4\Delta(G) - 4$ elements. This means $4\Delta(G) - 4 > \Delta(G)$, a contradiction. Hence $|V_{max}(T)| \leq 3$.

Theorem 2.11. *Let T be a tree with $|V_{max}(T)| = 3$. Then $\rho_e^o(T) = \Delta(T)$ if and only if $T \cong P_5$.*
Proof. Let $\rho_e^o(T) = \Delta(T)$ and $V_{max}(T) = \{u, v, w\}$. Then by Theorem 2.9, the distance between any two vertices in $V_{max}(T)$ is at most two. Suppose that there exist two pairs of vertices in $V_{max}(T)$, say $\{u, v\}$ and $\{v, w\}$ such that $d(u,v) = d(v,w) = 2$. Then again by Theorem 2.9, $d(u,w) \leq 2$ and so there exists a vertex $x \in V(T)$ such that x is adjacent to all the vertices in $V_{max}(T)$. Also, note that $\Delta(T) \geq 4$ as $deg(x) \geq 3$. Now, the set $N_e(u) \cup N_e(v) \cup N_e(w) \setminus \{xu, xv, xw\}$ forms an EOPS of T with $\rho_e^o(T) \geq 3\Delta(T) - 3$. But then $\Delta(T) = \rho_e^o(T) \geq 3(\Delta(T) - 1)$, a contradiction. So exactly one pair of vertices in $V_{max}(T)$ has distance equal to two and the other pairs have distance equal to one. Without loss of generality, let $d(u,w) = 2$ and $d(u,v) = d(v,w) = 1$. Note that the set $N_e(u) \cup N_e(w) \setminus \{uv, vw\}$ forms an EOPS. Clearly $\rho_e^o(T) \geq 2\Delta(T) - 2$ so that $\Delta(T) \leq 2$. If $\Delta(T) = 1$, then $T \cong K_2$. As $|V_{max}(T)| = 3$, it follows that $\Delta(T) = 2$ and so T is a path and by the Theorem 1.1, $T \cong P_5$. The converse follows from Theorem 1.1.

Definition 2.12. *Two pendent vertices v_1 and v_2 in a graph G are said to have a distinct support if $N(v_1) \cap N(v_2) = \phi$.*

Theorem 2.13. *Let T be a tree of order at least 2 with $|V_{max}(T)| = 2$. Then $\rho_e^o(T) = \Delta(T)$ if and only if T is isomorphic to either $B_{r,r}$ with $r \geq 0$ or T_1 or T_2, where T_1 is obtained from $B_{r,r}$ by adding exactly one pendant edge at any one of the pendant vertices of $B_{r,r}$ and T_2 is obtained from $B_{r,r}$ by adding exactly one pendant edge at exactly two pendant vertices in $B_{r,r}$ with distinct support.*
Proof. Let $\rho_e^o(T) = \Delta(T)$ and $|V_{max}(T)| = 2$. If $\Delta(T) = 1$, then $T \cong K_2$ and

if $\Delta(T) = 2$, then $T \cong P_4$. Now, let $\Delta(T) \geq 3$ with $V_{max}(T) = \{u, v\}$. Then by Theorem 2.9, $d(u, v) \leq 2$. Suppose that $d(u, v) = 2$. Then there exists a vertex $x \in V(T)$ which connects the vertices u and v by a path (u, x, v). Note that the set $N_e(u) \cup N_e(v) \setminus \{ux, xv\}$ forms an EOPS. Clearly $\rho_e^o(T) \geq 2\Delta(T) - 2$ implies $\Delta(T) \leq 2$, a contradiction to the fact that $\Delta(T) \geq 3$. So $d(u, v) = 1$. Now, all the edges incident at u(or v) and all the edges incident at $N_4(u)$(or $N_4(v)$) forms an EOPS of T. As $\rho_e^o(T) = \Delta(T)$ and $deg(u) = deg(v) = \Delta(T)$, we have $N_4(u) = N_4(v) = \phi$. This means that $d(u, x) \leq 3$ and $d(v, x) \leq 3$ for all $x \in V(T)$. As $deg(u) = \Delta(T)$, let $N(u) \setminus \{v\} = \{u_1, u_2, ..., u_{\Delta(T)-1}\}$. As the set $\{N_e(v) \cup N_e(u_i)\} \setminus \{uv, uu_i\}$ forms an EOPS for each $i(1 \leq i \leq \Delta(T) - 1)$, it follows that $\rho_e^o(T) \geq \Delta(T) - 1 + \sum_{i=1}^{\Delta(T)-1}(deg(u_i) - 1) = \Delta(T) - 1 + deg(u_1) + deg(u_2) + ... + deg(u_{\Delta(T)-1}) - (\Delta(T) - 1)$. This means that $\Delta(T) = \rho_e^o(T) \geq deg(u_1) + deg(u_2) + ... + deg(u_{\Delta(T)-1})$. So at most one vertex in $N(u) \setminus \{v\}$ or at most one vertex in $N(v) \setminus \{u\}$ has degree equal to two and rest of them have degree equal to one. This means that $T \cong B_{r,r}$ with $r \geq 2$ if every vertex in $N(u) \cup N(v) \setminus \{u, v\}$ has degree equal to one and $T \cong T_1$ if exactly one vertex in $N(u) \cup N(v) \setminus \{u, v\}$ has degree equal to two. Further, if exactly one vertex in $N(u) \setminus \{v\}$ and exactly one vertex in $N(v) \setminus \{u\}$ have degree equal to two, then $T \cong T_2$.

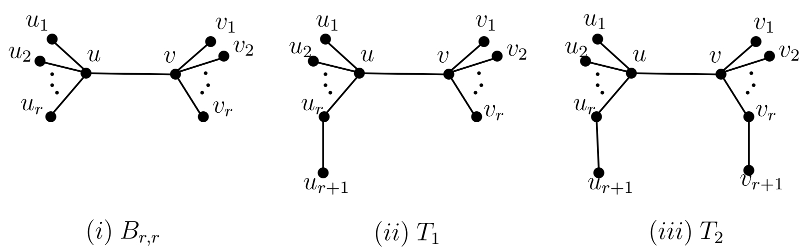

Figure 1: Trees with $\rho_e^o(T) = \Delta(T)$ when $|V_{max}(T)| = 2$.

Conversely, if T is isomorphic to $B_{r,r}$ for $r \geq 0$ or T_1 or T_2 as shown in Figure 1, then one can construct an EOPS with $\Delta(T)$ edges as follows. In the case of $B_{r,r}$, choose $S_1 = \{uu_i, uv\}$ or $S_1' = \{uv, vv_i\}$ for $1 \leq i \leq r$ and in the case of T_1, choose $S_2 = S_1$ or S_1' of $B_{r,r}$ or $S_3 = \{vv_i, u_r u_{r+1}\}$ for $1 \leq i \leq r$. In the case of T_2, choose $S_4 = S_2$ or $S_5 = \{uu_i, v_r v_{r+1}\}$ for $1 \leq i \leq r$. One can conclude by noting that $|S_1| = |S_1'| = |S_2| = |S_3| = |S_4| = |S_5| = r + 1 = \Delta(T) = \rho_e^o(T)$.

Theorem 2.14. *Let T be a tree. Then $\rho_e^o(T) = 2$ if and only if T is either P_3 or P_4 or P_5.*

Proof. Suppose that $\rho_e^o(T) = 2$. Let v be a vertex of T such that $deg(v) = \Delta(T)$. Since the edges that are incident at v forms an EOPS of T, it follows that $\Delta(T) \leq \rho_e^o(T) = 2$. If $\Delta(T) = 1$, then $T \cong K_2$, a contradiction and hence $\Delta(T) = 2$. Therefore, T is a path. By Proposition 1.1, T is either P_3 or P_4 or P_5. Further, the converse follows from Proposition 1.1.

Theorem 2.15. *Let T be a tree of order at least 4. Then $\rho_e^o(T) = 3$ if and only if either $T \cong P_6$ or $T \in \{B_{2,s} : 0 \leq s \leq 2\}$ or T is isomorphic to one of the graphs in $\{T_1, T_2, T_3, T_4, T_5, T_6\}$ shown in Figure 2.*

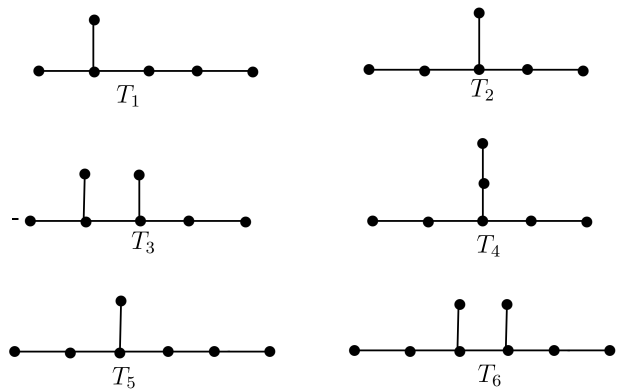

Figure 2: Trees T_1 to T_6.

Proof. Let $\rho_e^o(T) = 3$. Then by Proposition 2.6, $diam(T) \leq 5$ and by Observation 2.8(i), $\Delta(T) \leq 3$. If $\Delta(T) = 2$, then T is a path and by Theorem 1.1, $T \cong P_6$. We proceed further using induction on the diameter of T. Suppose that $\Delta(T) = 3$ and $diam(T) = 2$. Let $P_3 : (u_1u_2u_3)$ be a tree with diameter two. Clearly, P_3 must be an induced subgraph of such a tree T. One cannot extend P_3 by adding an edge at the pendant vertices of P_3. This is because, it will create in such an extension, a subtree with diameter 3, a contradiction. This means without violating the diameter 2, one cannot extend P_3 to a tree other than a bistar $B_{2,0}$. Similarly, if $\Delta(G) = 3$ and $diam(T) = 3$, then such a tree T will have P_4 as an induced subgraph. One can extend P_4 into a bigger subtree by only attaching an edge at any one of its two degree vertices as before. This means P_4 cannot be extended into a tree other than a bistar $B_{2,1}$ or $B_{2,2}$. Now among these bistars, $\rho_e^o(B_{2,0}) = \rho_e^o(B_{2,1}) = \rho_e^o(B_{2,2}) = 3$. Hence $T \in \{B_{2,s} : 0 \leq s \leq 2\}$.

Next, assume that $diam(T) = 4$. Then T will have $P_5 : (u_1u_2u_3u_4u_5)$ as an induced subgraph. As before one can extend P_5 into a subtree of T by adding edges at one of its two degree vertices in the following manner shown in Figure 2 labeled as T_1, T_2 ,T_3 and T_4.

Now, let $diam(T) = 5$. Then T will have $P_6 : (u_1u_2u_3u_4u_5u_6)$ as an induced subgraph. As before, we can extend P_6 into a subtree of T by adding edges at one of its two degree vertices, in the following manner shown in Figure 2 labeled as T_5, T_6. One can check in a routine manner that $\rho_e^o(T_j) = 3$ for $1 \leq j \leq 6$.

2.3. Unicyclic Graphs

Proposition 2.16. *For any unicyclic graph G, we have $\rho_e^o(G) \geq \Delta(G) - 1$.*

Proof. Let G be a unicyclic graph with cycle C. If $C \cong C_3 : (v_1v_2v_3v_1)$, then any EOPS of G can contain at most one edge from C_3. Also, if $v_i = \Delta(G)$ for some $i = 1, 2$ or 3, then any EOPS can contain at most $\Delta(G) - 1$ edges incident at v_i. Now, if some vertex $v \in V(G)\backslash V(C)$ happens to be a maximum degree vertex then an EOPS of G can contain all the edges incident with v. This means $\rho_e^o(G) \geq \Delta(G)$. Due to the observations, if all maximum degree vertices of G happens to be any one of the v_i for $i = 1, 2$ or 3 then $\rho_e^o(G) \geq \Delta(G) - 1$.

Proposition 2.17. *Let G be a unicyclic graph. If $H = \langle C_3 \rangle \subseteq G$ and $\rho_e^o(G) = 2$, then $\langle E(G) \setminus E(C) \rangle \cong P_2$ or P_3.*

Proof. The edge induced subgraph $[E(G) \setminus E(C)]$ is a forest. So such a forest will have P_l as an induced subgraph for some l. We claim that $l = 2$ or 3. Suppose not, and $l \geq 4$. If $l = 4$, then either the unique cycle C in G touches one of the pendant vertex of P_l or $V(C) \cap V(P_l) = \phi$. In the case of former, as $\mu([E(G) \setminus E(C)]) \geq 2$, one can construct an EOPS with two elements from P_l and one element from the other component, where $\mu(H)$ is the number of components in H. This means $\rho_e^o(G) \geq 3$.

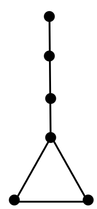

Figure 3: The Graph G^*.

The case that $G \cong G^*$ where G^* as shown in Figure 3 cannot arise as two edges from P_l and an edge of C non-incident with P_l together forms an EOPS with $\rho_e^o(G) \geq 3$. Now in the case of latter, one can easily construct an EOPS with two elements from P_l and any one edge from C, yielding $\rho_e^o(G) \geq 3$. This is a contradiction as $\rho_e^o(G) = 2$.

Note 2.18. *In the Proposition 2.17, the subgraph* $H = [E(G) \setminus E(C)]$ *contains at most one* P_3. *This is because, if* $G \cong G_1^*$, *where* G_1^* *shown in Figure 4, then* $diam(G) \geq 5$ *and* $\rho_e^o(G) \geq 3$. *Further if* $P_3 \subseteq H$, *then* H *cannot contain a* P_2. *This is because, if* $H \cong G_2^*$, *where* G_2^* *shown in Figure 5, then* $\rho_e^o(G) \geq 3$ *as one can form an EOPS with three elements.*

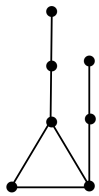

Figure 4: The Graph G_1^*.

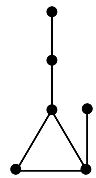

Figure 5: The Graph G_2^*.

Theorem 2.19. *Let* G *be a unicyclic graph. Then* $\rho_e^o(G) = 2$ *if and only if* G *is either* C_4, C_5 *or* C_6 *or isomorphic to one of the graphs* H_i, $(1 \leq i \leq 4)$ *shown in Figure 6.*

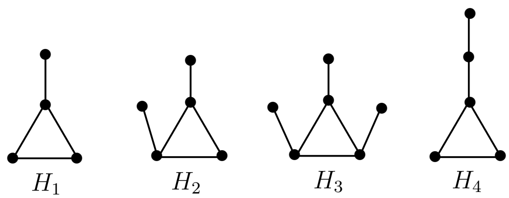

Figure 6: Graphs H_1 to H_4.

Proof. If G is a cycle, then the fact that G is one of the cycles C_4, C_5 or C_6 follows from Proposition 1.2.

Suppose that G is not a cycle. Let C be the cycle in G and l be its length. It is obvious that, if $l \geq 4$, then G has $K_{1,s}(s \geq 3)$ as an induced subgraph. So $\rho_e^o(G) \geq 3$, a contradiction. Thus $l = 3$ and $C = C_3$. As $\rho_e^o(G) = 2$ and $\rho_e^o(G) \geq \left\lceil \frac{diam(G)}{2} \right\rceil$, by Theorem 1.3, we deduce that $\left\lceil \frac{diam(G)}{2} \right\rceil \leq 2$ and hence $diam(G) \leq 4$. The following facts are easy to observe.

Fact 1. No vertex of C has degree greater than 3 as $\rho_e^o(G) = 2$.

Fact 2. At least one vertex of C has degree 3 as G is not a cycle.

Now, from Proposition 2.17 and Note 2.18, $\langle E(G) \setminus E(C) \rangle$ has at most one P_3. If $\langle E(G) \setminus E(C) \rangle$ has one P_3, then by Fact 1, Fact 2 and Note 2.18, $G \cong H_4$. Otherwise by Fact 1 and Fact 2, we have the following

(i) $G \cong H_1$ when exactly one vertex of C has degree 3 in G.

(ii) $G \cong H_2$ when exactly two vertices of C have degree 3 in G.

(iii) $G \cong H_3$ when all the vertices of C have degree 3 in G.

Conversely, Let G be C_4 or C_5 or C_6. Then from Proposition 1.2 that $\rho_e^o(C_4) = \left\lfloor \frac{4}{2} \right\rfloor = 2$, $\rho_e^o(C_5) = \left\lfloor \frac{5}{2} \right\rfloor = 2$ and $\rho_e^o(C_6) = \left\lfloor \frac{6}{2} \right\rfloor - 1 = 2$. Suppose G is isomorphic to one of the graphs in $\{H_i : 1 \leq i \leq 4\}$. Then we verify that the above listed graphs fulfills all the conditions in Theorem 1.3. As $diam(H_i) = 2$ and H_i is $K_{1,s}$-free($s \geq 3$) for all $1 \leq i \leq 4$, conditions (i) and (ii) given in Theorem 1.3 are true. Since the graph $H_i(1 \leq i \leq 3)$ has common edge for any two of its non-adjacent edges, there is no need to check the condition (iii). Consider H_4 with vertex labels as in Figure 6. Clearly H_4 consists of exactly one pair of non-adjacent edges having no common edge, let they be $e_1 = (u_1, u_2)$ and $e_2 = (v, w)$. Then $V(G) \setminus \{u_1, u_2, v, w\} = \{u\}$. As $deg(u) = 3$ in H_4, the condition (iii) is satisfied. Hence $\rho_e^o(G) = 2$ whenever G is one of the graphs listed in the statement.

3. Conclusion and Open Problems

The results obtained here opens a plethora of new open problems. As EOPN is a newly coined parameter, its relationship with other parameters such as packing number, total edge domination number of graphs, Nordhaus-Gaddum type extremal graph characterization tasks are yet to be taken up. That is, lower and upper bounds for $\rho_e^o(G) + \rho_e^o(G^c)$ and $\rho_e^o(G)\rho_e^o(G^c)$ are yet to be determined and certainly the investigation concerning their extremal graphs will be both difficult and interesting. The following concrete open problems are listed for the benefit of researchers in this area.

1. Obtain the sharp bounds for $\rho_e^o(T)$ of any tree T.

2. Obtain the extremal graphs for the equality $\rho_e^o(T) = \Delta(T)$, where $|V_{max}(T)| = 1$ and T is a tree.

3. Obtain the extremal graphs for the equality $\rho_e^o = \Delta(G) - 1$, where G is any unicyclic graph.

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